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Fitzpatrick transform of monotone relations in Hadamard spaces

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Abstract

In the present paper, monotone relations and maximal monotone relations from an Hadamard space to its linear dual space are investigated. Fitzpatrick transform of monotone relations in Hadamard spaces is introduced. It is shown that Fitzpatrick transform of a special class of monotone relations is proper, convex and lower semi-continuous. Finally, a representation result for monotone relations is given.

1 Introduction

The notion of spaces of non-positive curvature goes back to the work of J. Hadamard and E. Cartan in the 1920's. In 1950, H. Busemann and A.D. Aleksandrov have generalized the concept of geodesic metric space based on the concept of manifolds with the non-positive sectional curvature. Gromov, conceived the acronym CAT(0) for the non-positive curvature geodesic metric space, where, the letters C, A and T stand for Cartan, Aleksandrov and Toponogov, respectively. Moreover, 0 is the upper curvature bound. Also, Gromov developed and investigated many results about CAT(0) spaces.

A complete CAT(0) space is said to be an *Hadamard space*. Important examples of Hadamard spaces include the Hilbert spaces, Hadamard manifolds, $\mathbb{R}$-trees, Euclidean buildings, nonlinear Lebesgue spaces, Hilbert balls,

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complete simply connected Riemannian manifolds of non-positive sectional curvature, (see [8, Chapter II.1, 1.15] for many other examples).

Although Hadamard spaces have many properties of Hilbert spaces, such as weak convergence, metric projections onto closed and convex sets are nonexpansive mappings, the Kadek-Klee property, there are remarkable differences between the Hadamard and Hilbert spaces, for instance, a weak convergent sequence in a Hadamard space is not necessarily bounded.

In recent years, Hadamard spaces have become of strong interest in convex optimization, fixed point theory, ergodic theory, geometric group theory and many important applied topics in mathematics. For background materials on Hadamard spaces, we refer to the standard texts such as [8, 3, 4, 23].

This paper is organized as follows:

In Section 2, we collect some fundamental definitions and general notations of Hadamard spaces that will be used throughout of this paper. More precisely, geodesic path, geodesic space, CN-inequality, Cauchy-Schwarz inequality, quasilinearization of Hadamard spaces, dual and linear dual of an Hadamard space, also some facts regarding convexity in Hadamard spaces is given. In Section 3, flat Hadamard spaces are characterized. Indeed, it is shown that X is a flat Hadamard space if and only if $X \times X^\diamond$ has $\mathcal{P}$-property, where $X^\diamond$ is the linear dual of X. In addition, we prove that p-coupling function is sequentially $bw \times \|\cdot\|_\diamond$-continuous. Section 4 is devoted to investigate (maximal) monotone relations on Hadamard spaces. Some examples in flat and non-flat Hadamard spaces is illustrated. Finally, in Section 5, p-Fitzpatrick transform for subsets of $X \times X^\diamond$ is introduced. Then, some basic properties of this transform, specially for monotone relations, are considered. Also, we discuss the representation of monotone relations from X to $X^\diamond$, by proper, l.s.c. and convex functions on $X \times X^\diamond$.

2 Preliminaries

In this section, we collect some fundamental definitions and results on quasilinearization of Hadamard spaces that will be used throughout of this paper. For more details, we refer to [6, 2, 10].

Let (X, d) be a metric space. We say that a mapping $c : [0, 1] \to X$ is a *geodesic path* from $x \in X$ to $y \in X$ if $c(0) = x$, $c(1) = y$ and

$$d(c(t), c(s)) = |t - s|d(x, y),$$

for each $t, s \in [0, 1]$. The image of c is said to be a *geodesic segment* joining x and y. A metric space (X, d) is called a *geodesic space* if there is a geodesic path between every two points of X. Also, a geodesic space X is called *uniquely geodesic space* if for each $x, y \in X$ there exists a unique geodesic path from x

to y. From now on, in a uniquely geodesic space, we denote the set $c([0,1])$ by $[x,y]$ and for each $z\in[x,y]$, we write $z=(1-t)x\oplus ty$, where $t\in[0,1]$. In this case, we say that z is a *convex combination* of x and y. Hence, $[x,y]=\{(1-t)x\oplus ty : t\in[0,1]\}$. More details can be found in [3, 8].

A *geodesic triangle* $\Delta:=\Delta(x_1,x_2,x_3)\subseteq X$ consists of vertices $x_1,x_2,x_3\in X$ and geodesic paths $[x_1,x_2]$, $[x_2,x_3]$ and $[x_1,x_3]$. A *comparison triangles* for geodesic triangle $\Delta(x_1,x_2,x_3)\subseteq X$ is a triangle $\overline{\Delta}:=\Delta(\overline{x_1},\overline{x_2},\overline{x_3})\subseteq\mathbb{E}^2$ such that $d(x_i,x_j)=\|\overline{x_i}-\overline{x_j}\|_{\mathbb{E}^2}$, for $i,j\in\{1,2,3\}$. Now, let $a\in\Delta$, the *comparison point* for $a=(1-t)x_1\oplus tx_2\in[x_1,x_2]$ in $\overline{\Delta}$ is denoted by

$$\overline{a}:=(1-t)\overline{x_1}+t\overline{x_2}\in[\overline{x_1},\overline{x_2}].$$

We say that Δ has CAT(0) *inequality* if for each $a,b\in\Delta$ and each comparison points $\overline{a},\overline{b}\in\overline{\Delta}$,

$$d(a,b)\leq\|\overline{a}-\overline{b}\|_{\mathbb{E}^2}. \tag{1}$$

The geodesic space (X,d) is called CAT(0) *space* if all geodesic triangles satisfy the CAT(0) inequality (1). Hence, CAT(0) spaces are the special class of geodesic metric spaces in which geodesic triangles are thinner than its comparison triangle in the Euclidean plane. It is well known that (see [11, Proposition 5.1.9]) a geodesic space (X,d) is a CAT(0) space if and only if we have:

$$d(z,c(t))^2\leq(1-t)d(z,x)^2+td(z,y)^2-t(1-t)d(x,y)^2, \tag{2}$$

for each geodesic path $c:[0,1]\to X$ from x to y, each $z\in X$ and each $t\in[0,1]$. Inequality (2) is called *CN-inequality*. One can show that (for instance see [3, Theorem 1.3.3]) CAT(0) spaces are uniquely geodesic spaces.

In 2008, Berg and Nikolaev [6] introduced the concept of *quasilinearization* in abstract metric spaces. Ahmadi Kakavandi and Amini [2] defined the *dual space* for an Hadamard space (X,d) by using the concept of quasilinearization of X. More precisely, let X be an Hadamard space. For each $x,y\in X$, the ordered pair $(x,y)\in X^2$ is denoted by $\overrightarrow{xy}$ and will be called a *bound vector* in which x and y are called *tail* and *head* of $\overrightarrow{xy}$, respectively. For each $x\in X$, the *zero bound vector* at $x\in X$ will be written as $\mathbf{0}_x:=\overrightarrow{xx}$. We identify two bound vectors $-\overrightarrow{xy}$ and $\overrightarrow{yx}$. The bound vectors $\overrightarrow{xy}$ and $\overrightarrow{uz}$ are called *admissible* if $y=u$. The operation of addition of two admissible bound vectors $\overrightarrow{xy}$ and $\overrightarrow{yz}$ is defined by $\overrightarrow{xy}+\overrightarrow{yz}=\overrightarrow{xz}$. The *quasilinearization map* is defined by

$$\begin{aligned}&\langle\cdot,\cdot\rangle : X^2\times X^2\to\mathbb{R}, \\ &\langle\overrightarrow{ab},\overrightarrow{cd}\rangle:=\frac{1}{2}\big\{d(a,d)^2+d(b,c)^2-d(a,c)^2-d(b,d)^2\big\},\ a,b,c,d,\in X.\end{aligned} \tag{3}$$

For each $a,b,c,d,u\in X$ we have:

(i) $\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle = \langle \overrightarrow{cd}, \overrightarrow{ab} \rangle$,

(ii) $\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle = -\langle \overrightarrow{ba}, \overrightarrow{cd} \rangle$,

(iii) $\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle = \langle \overrightarrow{au}, \overrightarrow{cd} \rangle + \langle \overrightarrow{ub}, \overrightarrow{cd} \rangle$.

A metric space (X, d) satisfies the *Cauchy-Schwarz inequality* if for each $a, b, c, d \in X$ we have:

$$\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle \leq d(a,b)d(c,d). \tag{4}$$

Cauchy-Schwarz inequality is an alternative approach to characterization of the CAT(0) spaces. Indeed,

Theorem 2.1. [6, Corollary 3] *A geodesic metric space (X, d) is* CAT(0) *space if and only if it satisfies the Cauchy-Schwarz inequality.*

Now, in order to define the dual space of Hadamard space (X, d), consider the map

$$\begin{aligned} \Theta &: \mathbb{R} \times X^2 \to C(X, \mathbb{R}) \\ (t, a, b) &\mapsto \Theta(t, a, b)x = t\langle \overrightarrow{ab}, \overrightarrow{ax} \rangle, \ a, b, x \in X, \ t \in \mathbb{R}, \end{aligned}$$

where $C(X, \mathbb{R})$ is the space of all continuous real-valued functions on X. By using the Cauchy-Schwarz inequality (4), $\Theta(t, a, b)$ is a Lipschitz function with Lipschitz semi-norm

$$L(\Theta(t, a, b)) = |t|d(a, b), \ a, b \in X, \ t \in \mathbb{R}. \tag{5}$$

Recall that the *Lipschitz semi-norm* on $C(X, \mathbb{R})$ is defined by

$$\begin{aligned} L &: C(X, \mathbb{R}) \to \mathbb{R} \\ \varphi &\mapsto \sup \Big\{ \frac{\varphi(x) - \varphi(y)}{d(x, y)} : x, y \in X, x \neq y \Big\}. \end{aligned}$$

The Lipschitz semi-norm (5) induces a *pseudometric* D on $\mathbb{R} \times X^2$, which is defined by

$$D((t, a, b), (s, c, d)) = L(\Theta(t, a, b) - \Theta(s, c, d)), \ a, b, c, d \in X, \ t, s \in \mathbb{R}.$$

Now, the pseudometric space $(\mathbb{R} \times X^2, D)$ can be considered as a subspace of the pseudometric space of all real-valued Lipschitz functions $(\mathrm{Lip}(X, \mathbb{R}), L)$.

Lemma 2.2. [2, Lemma 2.1] *Let (X, d) be an Hadamard space. Then*

$$D((t, a, b), (s, c, d)) = 0 \text{ if and only if } t\langle \overrightarrow{ab}, \overrightarrow{xy} \rangle = s\langle \overrightarrow{cd}, \overrightarrow{xy} \rangle \text{ for all } x, y \in X.$$

It is easily seen that D induces an equivalence relation on $\mathbb{R} \times X^2$. Indeed, for each $(t, a, b) \in \mathbb{R} \times X^2$, the equivalence class of (t, a, b) is given by

$$[t\overrightarrow{ab}] = \{s\overrightarrow{cd} : D((t, a, b), (s, c, d)) = 0\}.$$

The set of all equivalence classes equipped with the metric D, defined by

$$D([t\overrightarrow{ab}], [s\overrightarrow{cd}]) := D((t, a, b), (s, c, d)),$$

is called the *dual space* of the Hadamard space X, and is denoted by X^*. By using the definition of equivalence classes, we get $[\overrightarrow{aa}] = [\overrightarrow{bb}]$ for each $a, b \in X$. In general X^* acts on X^2 by

$$\langle x^*, \overrightarrow{xy}\rangle := t\langle \overrightarrow{ab}, \overrightarrow{xy}\rangle, \text{ where } x^* = [t\overrightarrow{ab}] \in X^* \text{ and } \overrightarrow{xy} \in X^2.$$

Throughout this paper, we use the following notation:

$$\Big\langle \sum_{i=1}^{n} \alpha_i x_i^*, \overrightarrow{xy} \Big\rangle := \sum_{i=1}^{n} \alpha_i \langle x_i^*, \overrightarrow{xy}\rangle, \ \alpha_i \in \mathbb{R}, x_i^* \in X^*, \overrightarrow{xy} \in X^2, n \in \mathbb{N}.$$

In [10], Chaipunya and Kumam introduced the concept of *linear dual space* of Hadamard space X, as follows:

$$X^\diamond := \Big\{ \sum_{i=1}^{n} \alpha_i x_i^* : \alpha_i \in \mathbb{R}, x_i^* \in X^*, n \in \mathbb{N} \Big\}.$$

The zero element of $X^\diamond$ is denoted by $\mathbf{0}_{X^\diamond} := [t\overrightarrow{aa}]$, where $a \in X$ and $t \in \mathbb{R}$. One can see that the evaluation $\langle \mathbf{0}_{X^\diamond}, \cdot\rangle$ vanishes on X^2. It is worth mentioned that $X^\diamond$ is a normed space with the norm $\|x^\diamond\|_\diamond = L(x^\diamond)$, for all $x^\diamond \in X^\diamond$. Indeed:

Lemma 2.3. [24, Proposition 3.5] *Let X be an Hadamard space with linear dual space $X^\diamond$ and let $x^\diamond \in X^\diamond$ be arbitrary. Then*

$$\|x^\diamond\|_\diamond := \sup\Big\{ \frac{|\langle x^\diamond, \overrightarrow{ab}\rangle - \langle x^\diamond, \overrightarrow{cd}\rangle|}{d(a, b) + d(c, d)} : a, b, c, d \in X, (a, c) \neq (b, d) \Big\},$$

is a norm on $X^\diamond$. In particular, $\|[t\overrightarrow{ab}]\|_\diamond = |t|d(a, b)$.

Definition 2.4. [3, Section 2.2] Let (X, d) be an Hadamard space and let $f : X \to]-\infty, \infty]$ be a function. Then

(i) *domain* of f is defined by

$$\mathrm{dom} f = \{x \in X : f(x) < \infty\}.$$

Moreover, f is called *proper* if $\mathrm{dom} f \neq \emptyset$.

(ii) f is *lower semi-continuous* (briefly, *l.s.c.*) if for each $\alpha \in \mathbb{R}$, the set $\text{epi}f := \{(x, \alpha) \in X \times \mathbb{R} : f(x) \leq \alpha\}$ is closed.

(iii) f is said to be *convex* if

$$f((1-\lambda)x \oplus \lambda y) \leq (1-\lambda)f(x) + \lambda f(y),$$

where $x, y \in X$ and $\lambda \in [0, 1]$.

Moreover, f is called *concave* if $-f$ is convex.

The set of all proper, l.s.c. and convex extended real-valued functions on X is denoted by $\Gamma(X)$.

Definition 2.5. [2, Definition 2.4] Let $\{x_n\}$ be a sequence in an Hadamard space X. The sequence $\{x_n\}$ is said to be *weakly convergent* to $x \in X$ if $\lim_{n\to\infty}\langle \overrightarrow{xx_n}, \overrightarrow{xy}\rangle = 0$, for all $y \in X$. In this case we write, $x_n \xrightarrow{w} x$.

Lemma 2.6. [24, Proposition 3.6] *Let $\{x_n\}$ be a bounded sequence in an Hadamard space (X, d) with linear dual space $X^{\diamond}$ and let $\{x_n^{\diamond}\}$ be a sequence in $X^{\diamond}$. If $x_n \xrightarrow{w} x$ and $x_n^{\diamond} \xrightarrow{\|\cdot\|_{\diamond}} x^{\diamond}$, then $\langle x_n^{\diamond}, \overrightarrow{x_n z}\rangle \to \langle x^{\diamond}, \overrightarrow{xz}\rangle$, for all $z \in X$.*

It is well known that convergence in the metric implies weak convergence.

3 Flat Hadamard spaces

Let X be an Hadamard space with linear dual space $X^{\diamond}$ and $M \subseteq X \times X^{\diamond}$. The *domain* and *range* of M are defined, respectively, by

$$\text{Dom}(M) := \{x \in X : \exists\, x^{\diamond} \in X^{\diamond} \text{ s.t. } (x, x^{\diamond}) \in M\},$$

and

$$\text{Range}(M) := \{x^{\diamond} \in X^{\diamond} : \exists\, x \in X \text{ s.t. } (x, x^{\diamond}) \in M\}.$$

Definition 3.1. [22, Definition 3.1] An Hadamard space (X, d) is said to be *flat* if equality holds in the CN-inequality, i.e., for each $x, y \in X$ and $\lambda \in [0, 1]$, the following holds:

$$d(z, (1-\lambda)x \oplus \lambda y)^2 = (1-\lambda)d(z, x)^2 + \lambda d(z, y)^2 - \lambda(1-\lambda)d(x, y)^2, \quad z \in X.$$

Definition 3.2. Let X be an Hadamard space with linear dual space $X^{\diamond}$ and $p \in X$ be fixed. We say that $M \subseteq X \times X^{\diamond}$ satisfies

(i) $\mathcal{F}_l$-property if for each $\lambda \in [0, 1]$, $x^{\diamond} \in \text{Range}(M)$ and $x, y \in \text{Dom}(M)$;

$$\left\langle x^{\diamond}, \overrightarrow{p((1-\lambda)x \oplus \lambda y)}\right\rangle \leq (1-\lambda)\langle x^{\diamond}, \overrightarrow{px}\rangle + \lambda\langle x^{\diamond}, \overrightarrow{py}\rangle. \tag{6}$$

(ii) $\mathcal{F}_g$-property if for each $\lambda \in [0,1]$, $x^\diamond \in \mathrm{Range}(M)$ and $x, y \in \mathrm{Dom}(M)$;

$$\big\langle x^\diamond, \overrightarrow{p((1-\lambda)x \oplus \lambda y)}\big\rangle \geq (1-\lambda)\langle x^\diamond, \overrightarrow{px}\rangle + \lambda\langle x^\diamond, \overrightarrow{py}\rangle. \tag{7}$$

(iii) $\mathcal{F}$-property if it has both $\mathcal{F}_l$ and $\mathcal{F}_g$-property.

Throughout this paper, let $\mathcal{P} \in \{\mathcal{F}, \mathcal{F}_l, \mathcal{F}_g\}$ and

$$\overline{\mathcal{P}} := \begin{cases} \mathcal{F} & if\ \mathcal{P} = \mathcal{F}, \\ \mathcal{F}_g & if\ \mathcal{P} = \mathcal{F}_l, \\ \mathcal{F}_l & if\ \mathcal{P} = \mathcal{F}_g. \end{cases}$$

Remark 3.3. *Note that*

(i) $\mathcal{F}_l$-property is investigated in [21], also it is introduced in [20] as $\mathcal{W}$-property.

(ii) One can see that $\mathcal{P}$-property is independent of the choice of $p \in X$. For instance, let $M \subseteq X \times X^\diamond$ satisfies in $\mathcal{F}_g$-property for some $p \in X$. Then for each $\lambda \in [0,1]$, $x^\diamond \in \mathrm{Range}(M)$ and $x, y \in \mathrm{Dom}(M)$, we have:

$$\big\langle x^\diamond, \overrightarrow{p((1-\lambda)x \oplus \lambda y)}\big\rangle \geq \langle x^\diamond, \overrightarrow{px}\rangle + \lambda(\langle x^\diamond, \overrightarrow{py}\rangle - \langle x^\diamond, \overrightarrow{px}\rangle).$$

Therefore,

$$\big\langle x^\diamond, \overrightarrow{p((1-\lambda)x \oplus \lambda y)} - \overrightarrow{px}\big\rangle \geq \lambda\langle x^\diamond, \overrightarrow{py} - \overrightarrow{px}\rangle,$$

and hence we conclude that $\langle x^\diamond, \overrightarrow{x((1-\lambda)x \oplus \lambda y)}\rangle \geq \lambda\langle x^\diamond, \overrightarrow{xy}\rangle$.

(iii) If $M \subseteq X \times X^\diamond$ has $\mathcal{P}$-property, then any subset of M has this property too.

(iv) $M \subseteq X \times X^\diamond$ has $\mathcal{P}$-property if and only if $M_{\diamondsuit}$ has $\overline{\mathcal{P}}$-property, where

$$M_{\diamondsuit} := \{(x, -x^\diamond) : (x, x^\diamond) \in M\}.$$

Lemma 3.4. [22, Theorem 3.2] *Let X be an Hadamard space. Then X is flat if and only if for each $x, y \in X$ and $\lambda \in [0,1]$:*

$$\langle \overrightarrow{x((1-\lambda)x \oplus \lambda y)}, \overrightarrow{ab}\rangle = \lambda\langle \overrightarrow{xy}, \overrightarrow{ab}\rangle,\ \ a, b \in X. \tag{8}$$

Theorem 3.5. *The following statements for an Hadamard space X are equivalent.*

(i) *X is flat.*

(ii) *$X \times X^{\diamond}$ has $\mathcal{P}$-property.*

(iii) *Any subset of $X \times X^{\diamond}$ has $\mathcal{P}$-property.*

Proof. Accordance with Remark 3.3(iii)&(iv) and the fact that $\left(X \times X^{\diamond}\right)_{\hat{\diamond}} = X \times X^{\diamond}$, it suffices to consider only the case that $\mathcal{P} = \mathcal{F}_l$. Indeed, we prove that X is flat if and only if $X \times X^{\diamond}$ has $\mathcal{F}_l$-property. Let $x, y \in X, \lambda \in [0,1]$ and $x^\diamond \in X^{\diamond}$. Then $x^\diamond = \sum_{i=1}^n \alpha_i [t_i \overrightarrow{a_i b_i}]$, where $n \in \mathbb{N}$, $1 \leq i \leq n$, $\alpha_i, t_i \in \mathbb{R}$ and $a_i, b_i \in X$. Hence, by using Lemma 3.4, we get:

$$\begin{aligned}
\langle x^\diamond, \overrightarrow{x((1-\lambda)x \oplus \lambda y)} \rangle &= \Big\langle \sum_{i=1}^n \alpha_i [t_i \overrightarrow{a_i b_i}], \overrightarrow{x((1-\lambda)x \oplus \lambda y)} \Big\rangle \\
&= \sum_{i=1}^n \alpha_i t_i \langle \overrightarrow{a_i b_i}, \overrightarrow{x((1-\lambda)x \oplus \lambda y)} \rangle \\
&\leq \lambda \sum_{i=1}^n \alpha_i t_i \langle \overrightarrow{a_i b_i}, \overrightarrow{xy} \rangle \\
&= \lambda \langle x^\diamond, \overrightarrow{xy} \rangle.
\end{aligned}$$

Therefore $X \times X^{\diamond}$ has $\mathcal{F}_l$-property. For the converse, it is sufficient to consider $p := x$ and $x^\diamond := \pm[\overrightarrow{ab}]$ in (6) and apply Lemma 3.4. □

Example 3.6. *Let (X, d) be an Hadamard space with linear dual space $X^{\diamond}$. As a trivial example, any singleton set $\{(x, x^\diamond)\} \subseteq X \times X^{\diamond}$ has $\mathcal{F}$-property. For a non-trivial example, consider*

$$M = \Big\{ \big(x, x^\diamond = [\alpha \overrightarrow{xc(\mu)}]\big) : x, y \in X,\ c(\mu) = (1-\mu)x \oplus \mu y, \mu \in [0,1], \alpha \in \mathbb{R} \Big\}.$$

Then

$$\begin{aligned}
\langle x^\diamond, \overrightarrow{xc(\lambda)} \rangle - \lambda \langle x^\diamond, \overrightarrow{xy} \rangle &= \Big\langle [\alpha \overrightarrow{xc(\mu)}], \overrightarrow{xc(\lambda)} \Big\rangle - \lambda \Big\langle [\alpha \overrightarrow{xc(\mu)}], \overrightarrow{xy} \Big\rangle \\
&= \alpha \Big\langle \overrightarrow{xc(\mu)}, \overrightarrow{xc(\lambda)} \Big\rangle - \lambda \alpha \Big\langle \overrightarrow{xc(\mu)}, \overrightarrow{xy} \Big\rangle \\
&= \frac{1}{2} \alpha \Big(\big(d(x, c(\lambda))^2 + d(x, c(\mu))^2 - d(c(\lambda), c(\mu))^2\big) \\
&\quad - \lambda \big(d(x,y)^2 + d(c(\mu), x)^2 - d(c(\mu), y)^2\big) \Big) \\
&= \frac{1}{2} \alpha d(x,y)^2 \Big(\lambda^2 + \mu^2 - (\lambda - \mu)^2 - \lambda \big(1 + \mu^2 - (1-\mu)^2\big) \Big) \\
&= 0.
\end{aligned}$$

Therefore, $\langle x^\diamond, \overrightarrow{xc(\lambda)} \rangle = \lambda \langle x^\diamond, \overrightarrow{xy} \rangle$. Hence, M has $\mathcal{F}$-property.

The next example shows that there exists a relation $M \subseteq X \times X^{\diamond}$ in the non-flat Hadamard spaces which doesn't have the $\mathcal{P}$-property.

Example 3.7. Consider the following equivalence relation on $\mathbb{N} \times [0,1]$:

$$(n,t) \sim (m,s) \Leftrightarrow t = s = 0 \text{ or } (n,t) = (m,s).$$

Set $X := \frac{\mathbb{N}\times[0,1]}{\sim}$ and let $d : X \times X \to \mathbb{R}$ be defined by

$$d([(n,t)],[(m,s)]) = \begin{cases} |t-s| & n = m, \\ t+s & n \neq m. \end{cases}$$

The geodesic joining $x = [(n,t)]$ to $y = [(m,s)]$ is defined as follows:

$$(1-\lambda)x \oplus \lambda y := \begin{cases} [(n,(1-\lambda)t - \lambda s)] & 0 \leq \lambda \leq \frac{t}{t+s}, \\ [(m,(\lambda-1)t + \lambda s)] & \frac{t}{t+s} \leq \lambda \leq 1, \end{cases}$$

whenever $x \neq y$ and vacuously $(1-\lambda)x \oplus \lambda x := x$ for each $\lambda \in [0,1]$. It is known that (see [1, Example 4.7]) (X,d) is an $\mathbb{R}$-tree space. It follows from [3, Example 1.2.10], that $\mathbb{R}$-tree spaces are Hadamard space. For each $n \in \mathbb{N}$, set $x_n := [(n,\frac{1}{2})]$ and $y_n := [(n,\frac{1}{n})]$. Set $M := \big\{(x_n, [\overrightarrow{y_{n+1}y_n}]) : n \in \mathbb{N}\big\} \subseteq X \times X^{\diamond}$. Let $\lambda = \frac{3}{4}$ and $x^{\diamond} = \big[\overrightarrow{y_3y_2}\big]$. Then

$$(1-\lambda)x_4 \oplus \lambda y_2 = [(2,\frac{1}{4})],$$

and so

$$\big\langle x^{\diamond}, \overrightarrow{x_4((1-\lambda)x_4 \oplus \lambda y_2)}\big\rangle = \frac{7}{24},$$

while, $\lambda\langle x^{\diamond}, \overrightarrow{x_4y_2}\rangle = \frac{3}{8}$. Hence, M doesn't have the $\mathcal{F}_g$-property. On the other hand, by choosing $y^{\diamond} = \big[\overrightarrow{y_4y_3}\big]$ we get:

$$\big\langle y^{\diamond}, \overrightarrow{x_4((1-\lambda)x_4 \oplus \lambda y_2)}\big\rangle = \frac{13}{48},$$

and $\lambda\langle y^{\diamond}, \overrightarrow{x_4y_2}\rangle = \frac{3}{16}$. Then M doesn't have the $\mathcal{F}_l$-property. Hence, by definition, M doesn't satisfy in $\mathcal{F}$-property and hence by Theorem 3.5, X is not flat.

Example 3.8. *Let X be the same as in Example 3.7. Set*

$$M := \Big\{\Big([(1,t)], \big[\overrightarrow{[(1,0)][(1,t)]}\big]\Big) : t \in [0,1]\Big\}.$$

Let $x^\diamond = \left[\overrightarrow{[(1,0)][(1,r)]}\right] \in \mathrm{Range}(M)$*,* $t, s \in [0,1]$ *with* $t \neq s$*,* $x = [(1,t)]$ *and* $y = [(1,s)]$*. Then* $x, y \in \mathrm{Dom}(M)$ *and for each* $\lambda \in [0,1]$ *we have:*

$$(1-\lambda)x \oplus \lambda y := \begin{cases} [(1,(1-\lambda)t - \lambda s)] & 0 \leq \lambda \leq \frac{t}{t+s}, \\ [(1,(\lambda-1)t + \lambda s)] & \frac{t}{t+s} \leq \lambda \leq 1, \end{cases} \tag{9}$$

and

$$\left\langle x^\diamond, \overrightarrow{x((1-\lambda)x \oplus \lambda y)} \right\rangle = \begin{cases} -r\lambda(t+s) & 0 \leq \lambda \leq \frac{t}{t+s}, \\ r(\lambda(t+s) - 2t) & \frac{t}{t+s} \leq \lambda \leq 1. \end{cases} \tag{10}$$

On the other hand,

$$\lambda \langle x^\diamond, \overrightarrow{xy} \rangle = r\lambda(s-t). \tag{11}$$

We observe that

$$\left\langle x^\diamond, \overrightarrow{x((1-\lambda)x \oplus \lambda y)} \right\rangle \leq \lambda \langle x^\diamond, \overrightarrow{xy} \rangle,$$

thus M *has* $\mathcal{F}_l$*-property. Moreover, let* $x^\diamond = \left[\overrightarrow{[(1,0)][(1,\frac{1}{3})]}\right]$*,* $x = [(1,1)]$*,* $y = [(1,\frac{1}{2})]$ *and* $\lambda = \frac{3}{5}$*. By using* (9)*,* (10) *and* (11) *we obtain:*

$$(1-\lambda)x \oplus \lambda y = [(1, \frac{1}{10})],$$

$$\left\langle x^\diamond, \overrightarrow{x((1-\lambda)x \oplus \lambda y)} \right\rangle = -\frac{3}{10},$$

and

$$\lambda \langle x^\diamond, \overrightarrow{xy} \rangle = -\frac{1}{10}.$$

Thus M *doesn't have* $\mathcal{F}_g$*-property. It follows from Remark 3.3(iv) that*

$$M_{\diamondsuit} = \left\{ \left([(1,t)], \left[\overrightarrow{[(1,t)][(1,0)]}\right] \right) : t \in [0,1] \right\},$$

has $\mathcal{F}_g$*-property but doesn't possess* $\mathcal{F}_l$*-property.*

Definition 3.9. Let X be an Hadamard space with linear dual space $X^\diamond$.

(i) A sequence $\{x_n\} \subseteq X$ is *bw-convergent* to $x \in X$, if $\{x_n\}$ is bounded and $x_n \xrightarrow{w} x$. In this case, we write $x_n \xrightarrow{bw} x$.

(ii) A sequence $\{(x_n, x_n^\diamond)\} \subseteq X \times X^\diamond$ is called $bw \times \|\cdot\|_\diamond$*-convergent* to $(x, x^\diamond) \in X \times X^\diamond$, if $x_n \xrightarrow{bw} x$ and $x_n^\diamond \xrightarrow{\|\cdot\|_\diamond} x^\diamond$. In this case, we write $(x_n, x_n^\diamond) \xrightarrow{bw \times \|\cdot\|_\diamond} (x, x^\diamond)$.

(iii) The mapping $\varphi : X \times X^{\diamond} \to]-\infty, \infty]$ is said to be *sequentially* $bw \times \|\cdot\|_{\diamond}$-*continuous at* $(x, x^{\diamond}) \in X \times X^{\diamond}$ if for every $\{(x_n, x_n^{\diamond})\} \subseteq X \times X^{\diamond}$, with $(x_n, x_n^{\diamond}) \xrightarrow{bw \times \|\cdot\|_{\diamond}} (x, x^{\diamond})$ we have $\varphi(x_n, x_n^{\diamond}) \to \varphi(x, x^{\diamond})$. Moreover, φ is *sequentially* $bw \times \|\cdot\|_{\diamond}$-*continuous* if it is sequentially $bw \times \|\cdot\|_{\diamond}$-continuous at each point of $X \times X^{\diamond}$.

Definition 3.10. For an arbitrary and fixed element $p \in X$, we define the *p-coupling function* of the dual pair $(X, X^{\diamond})$ as follows:

$$\pi_p : X \times X^{\diamond} \to \mathbb{R}; (x, x^{\diamond}) \mapsto \langle x^{\diamond}, \overrightarrow{px} \rangle.$$

This function is useful in the formulation of some basic results of the monotone relations in Hadamard spaces. Some properties of p-coupling function is considered in the following Lemma.

Lemma 3.11. *Suppose X is an Hadamard space with linear dual space $X^{\diamond}$, $M \subseteq X \times X^{\diamond}$ and $p \in X$. Then π_p is sequentially $bw \times \|\cdot\|_{\diamond}$-continuous and hence continuous.*

Proof. Let $\{(x_n, x_n^{\diamond})\} \subseteq X \times X^{\diamond}$ be such that $(x_n, x_n^{\diamond}) \xrightarrow{bw \times \|\cdot\|_{\diamond}} (x, x^{\diamond})$, where $(x, x^{\diamond}) \in X \times X^{\diamond}$. It follows from Lemma 2.6 that

$$\langle x_n^{\diamond}, \overrightarrow{px_n} \rangle \to \langle x^{\diamond}, \overrightarrow{px} \rangle,$$

which implies that π_p is sequentially $bw \times \|\cdot\|_{\diamond}$-continuous. Moreover, since convergence in the metric implies weak convergence, we conclude that π_p is continuous. $\square$

Lemma 3.12. *Suppose X is an Hadamard space with linear dual space $X^{\diamond}$, $M \subseteq X \times X^{\diamond}$ and $p \in X$. Then the following hold:*

(i) If M has the $\mathcal{F}_l$-property, then π_p is convex with respect to its first variable on M; in the sense that, for each $x, y \in \mathrm{Dom}(M)$, each $\lambda \in [0, 1]$ and each $x^{\diamond} \in \mathrm{Range}(M)$,

$$\pi_p\big((1-\lambda)x \oplus \lambda y, x^{\diamond}\big) \leq (1-\lambda)\pi_p(x, x^{\diamond}) + \lambda \pi_p(y, x^{\diamond}).$$

(ii) If M satisfies the $\mathcal{F}_g$-property, then π_p is concave with respect to its first variable on M, where concavity has a similar interpretation to that of convexity.

(iii) If M has the $\mathcal{F}$-property, then π_p is an affine mapping with respect to its first variable on M, in the sense that, for each $x, y \in \mathrm{Dom}(M)$, each $\lambda \in [0, 1]$ and each $x^{\diamond} \in \mathrm{Range}(M)$;

$$\pi_p\big((1-\lambda)x \oplus \lambda y, x^{\diamond}\big) = (1-\lambda)\pi_p(x, x^{\diamond}) + \lambda \pi_p(y, x^{\diamond}).$$

(iv) π_p is linear with respect to it's second variable on $X \times X^{\diamond}$.

Proof. (i): Let $x^\diamond \in \mathrm{Range}(M)$, $x, y \in \mathrm{Dom}(M)$ and $\lambda \in [0,1]$. By $\mathcal{F}_l$-property of M, we get:

$$\begin{aligned}\pi_p\big((1-\lambda)x \oplus \lambda y, x^\diamond\big) &= \langle x^\diamond, \overrightarrow{p((1-\lambda)x \oplus \lambda y)}\rangle \\ &\leq (1-\lambda)\langle x^\diamond, \overrightarrow{px}\rangle + \lambda\langle x^\diamond, \overrightarrow{py}\rangle \\ &= (1-\lambda)\pi_p(x, x^\diamond) + \lambda\pi_p(y, x^\diamond).\end{aligned}$$

(ii): It follows from Remark 3.3(iv) that $M_{\diamondminus}$ has $\mathcal{F}_l$-property. Hence, π_p is convex with respect to its first variable on $M_{\diamondminus}$. Therefore, π_p is concave with respect to its first variable on M.

(iii): Let M has $\mathcal{F}$-property. Since by Definition 3.2(iii), M has both of $\mathcal{F}_g$ and $\mathcal{F}_l$ properties, we get π_p is concave and convex on M, respectively. Consequently, π_p is an affine mapping on M.

(iv): Let $x^\diamond, y^\diamond \in X^{\diamond}$, $x \in X$ and $\alpha, \beta \in \mathbb{R}$. Then

$$\begin{aligned}\pi_p\big(x, \alpha x^\diamond + \beta y^\diamond\big) &= \langle \alpha x^\diamond + \beta y^\diamond, \overrightarrow{px}\rangle \\ &= \alpha\langle x^\diamond, \overrightarrow{px}\rangle + \beta\langle y^\diamond, \overrightarrow{px}\rangle \\ &= \alpha\pi_p(x, x^\diamond) + \beta\pi_p(a, y^\diamond),\end{aligned}$$

and the result follows.

4 Monotone Relations

Ahmadi Kakavandi and Amini [2] introduced the notion of monotone operators in Hadamard spaces. In [14], Khatibzadeh and Ranjbar, investigated some properties of monotone operators and their resolvents and also proximal point algorithm in Hadamard spaces. Chaipunya and Kumam [10] studied the general proximal point method for finding a zero point of a maximal monotone set-valued vector field defined on Hadamard spaces with valued in its linear dual. They proved the relation between the maximality and Minty's surjectivity condition. Zamani Eskandani and Raeisi [24], by using products of finitely many resolvents of monotone operators, proposed an iterative algorithm for finding a common zero of a finite family of monotone operators and a common fixed point of an infinitely countable family of non-expansive mappings in Hadamard spaces.

Definition 4.1. We say that $(x, x^\diamond) \in X \times X^{\diamond}$ and $(y, y^\diamond) \in X \times X^{\diamond}$ are *monotonically related*, if $\langle x^\diamond - y^\diamond, \overrightarrow{yx}\rangle \geq 0$ and it is denoted by $(x, x^\diamond)\mu(y, y^\diamond)$.

It is easy to see that μ is a reflexive and symmetric relation on $X \times X^{\diamond}$. Moreover, $(x, x^{\diamond}) \in X \times X^{\diamond}$ is monotonically related to M if

$$(x, x^{\diamond})\mu(y, y^{\diamond}), \text{for each } (y, y^{\diamond}) \in M,$$

and this is denoted by $(x, x^{\diamond})\mu M$. The *monotone polar* of M is defined by

$$M^{\mu} := \{(x, x^{\diamond}) \in X \times X^{\diamond} : (x, x^{\diamond})\mu M\}.$$

A relation $M \subseteq X \times X^{\diamond}$ is called *monotone* if every $(x, x^{\diamond})$, $(y, y^{\diamond}) \in M$ are monotonically related.

Definition 4.2. Let X be an Hadamard space with linear dual space $X^{\diamond}$. A monotone relation $M \subseteq X \times X^{\diamond}$ is called *maximal* if there is no monotone relation $N \subseteq X \times X^{\diamond}$ that contains properly M.

In other words, a monotone relation M is maximal if M contains all its monotonically related elements of $X \times X^{\diamond}$. The set of all maximal monotone relations $M \subseteq X \times X^{\diamond}$ is denoted by $\mathfrak{M}(X)$.

Remark 4.3. Similar to Theorem 20.21 of [5], using Zorn's Lemma, one can deduce that any monotone relation in Hadamard spaces can be extended to a maximal monotone relation. In other words, for each monotone relation $M \subseteq X \times X^{\diamond}$, there exists a maximal monotone relation $\widetilde{M}$ such that $M \subseteq \widetilde{M}$.

Proposition 4.4. *Let X be an Hadamard space and $M \subseteq X \times X^{\diamond}$. Then*

(i) M is monotone if and only if $M \subseteq M^{\mu}$.

(ii) $M \in \mathfrak{M}(X)$ if and only if $M = M^{\mu}$.

Proof. (i): A direct consequence of Definition 4.1.

(ii): Let $M \in \mathfrak{M}(X)$. It follows from (i) that $M \subseteq M^{\mu}$. On the other hand, let $(x, x^{\diamond}) \in M^{\mu}$. Hence, $(x, x^{\diamond})$ is monotonically related to M. Maximality of M implies that $(x, x^{\diamond}) \in M$. Conversely, suppose that $M = M^{\mu}$. Then, by (i), M is monotone. Moreover, let $(x, x^{\diamond}) \in X \times X^{\diamond}$ be such that $(x, x^{\diamond})\mu M$, therefore $(x, x^{\diamond}) \in M^{\mu} = M$. Thus M is a maximal monotone relation.

Example 4.5. *Let M be the same as in Example* 3.8. Then M is monotone. To see this, consider

$$\big(x = [(1,t)], x^{\diamond} = \big[\overrightarrow{[(1,0)][(1,t)]}\big]\big), \big(y = [(1,s)], y^{\diamond} = \big[\overrightarrow{[(1,0)][(1,s)]}\big]\big) \in M.$$

Therefore,

$$\langle x^{\diamond} - y^{\diamond}, \overrightarrow{yx}\rangle = (t-s)^2 \geq 0.$$

Moreover, M is not maximal. Take $(y = [(1,s)], y^\diamond = [\overrightarrow{(1,s)(1,0)}]) \in X \times X^\diamond$ and $\big(x = [(1,t)],\, x^\diamond = \big[\overrightarrow{[(1,0)][(1,t)]}\big]\big) \in M$. Then, by simple calculations,

$$\langle x^\diamond - y^\diamond, \overrightarrow{yx}\rangle = t^2 - s^2.$$

Now, take $0 < s < t$. Then $(y, y^\diamond)\mu M$ while $(y, y^\diamond) \notin M$. It follows from Proposition 4.4(ii) that M is not a maximal monotone relation.

Example 4.6. *Let (X, d) be a flat Hadamard space with linear dual space $X^\diamond$ and $f \in \Gamma(X)$. Define the mapping $\mathbb{I}_f : X \times X^\diamond \times X^\diamond \to]-\infty, \infty]$ by $\mathbb{I}_f(x, x^\diamond, y^\diamond) = \inf_{y \in X} \big\{ f(y) + \pi_y(x, x^\diamond + y^\diamond)\big\}$. For any $y^\diamond \in X^\diamond$, set*

$$M^f_{y^\diamond} := \big\{(x, x^\diamond) \in X \times X^\diamond : \mathbb{I}_f(x, x^\diamond, y^\diamond) \geq f(x)\big\}.$$

It is shown that $M^f_{y^\diamond}$ is a maximal monotone relation (see [21] for details).

5 Fitzpatrick transform

Krauss [15, 16, 17] represented maximal monotone operators by subdifferentials of skew-symmetric saddle functions on $E \times E$. Motivated by these works, Fitzpatrick [13] suggested a non-trivial way to represent maximal monotone operators by subdifferentials of convex functions on $E \times E^*$, where E is a reflexive Banach space and E^* is its dual Banach space. Martinez-Legaz and Théra [19] and Burachik and Svaiter [9], individually, rediscovered Fitzpatrick transform. Some of recent advantages on monotone operator theory and their Fitzpatrick transform can be found in [7, 12, 18] and the references cited therein.

In this section, we define Fitzpatrick transform for subsets of $X \times X^\diamond$. Then, some basic properties of Fitzpatrick transform, specially in the case that M is monotone, are investigated. Also, we discuss the representation of monotone relations from X to $X^\diamond$, by proper, l.s.c. and convex functions on $X \times X^\diamond$.

Let $h : X \times X^\diamond \to \overline{\mathbb{R}} := [-\infty, +\infty]$ be a function. We say that h is *convex* if for each $(x, x^\diamond)$, $(y, y^\diamond) \in X \times X^\diamond$ and each $\lambda \in [0,1]$:

$$h((1-\lambda)x \oplus \lambda y, (1-\lambda)x^\diamond + \lambda y^\diamond) \leq (1-\lambda)h(x, x^\diamond) + \lambda h(y, y^\diamond),$$

and h is *proper* if $-\infty \notin h(X \times X^\diamond)$ and $h \not\equiv \infty$. In the sequel, we denote the set of all extended real-valued convex lower semi-continuous and proper functions on $X \times X^\diamond$ by $\Gamma(X \times X^\diamond)$.

Given any $h : X \times X^{\diamond} \to \mathbb{R}$ and $p \in X$, set

$$\{h \leq \pi_p\} := \{(x, x^{\diamond}) \in X \times X^{\diamond} : h(x, x^{\diamond}) \leq \pi_p(x, x^{\diamond})\},$$
$$\{h < \pi_p\} := \{(x, x^{\diamond}) \in X \times X^{\diamond} : h(x, x^{\diamond}) < \pi_p(x, x^{\diamond})\},$$
$$\{h \geq \pi_p\} := \{(x, x^{\diamond}) \in X \times X^{\diamond} : h(x, x^{\diamond}) \geq \pi_p(x, x^{\diamond})\},$$
$$\{h > \pi_p\} := \{(x, x^{\diamond}) \in X \times X^{\diamond} : h(x, x^{\diamond}) > \pi_p(x, x^{\diamond})\},$$
$$\{h = \pi_p\} := \{(x, x^{\diamond}) \in X \times X^{\diamond} : h(x, x^{\diamond}) = \pi_p(x, x^{\diamond})\}.$$

Obviously,
(i) $\{h = \pi_p\} = \{h \leq \pi_p\} \cap \{h \geq \pi_p\}$,
(ii) $\{h < \pi_p\} = \{h \leq \pi_p\} \setminus \{h = \pi_p\}$,
(iii) $\{h > \pi_p\} = \{h \geq \pi_p\} \setminus \{h = \pi_p\}$.

Definition 5.1. Let X be an Hadamard space, $M \subseteq X \times X^{\diamond}$ and p be an arbitrary and fixed element of X. We define the *p-Fitzpatrick transform* of M as follows:

$$\begin{aligned}\Phi_M^p &: X \times X^{\diamond} \to [-\infty, \infty]\\ &(x, x^{\diamond}) \mapsto \sup_{(y,y^{\diamond}) \in M} \big\{\langle x^{\diamond}, \overrightarrow{py}\rangle - \langle y^{\diamond}, \overrightarrow{xy}\rangle\big\}.\end{aligned}$$

Note that, $\Phi_M^p \equiv -\infty$ if $M = \emptyset$; in fact we use the convention $\sup \emptyset = -\infty$. If $M \neq \emptyset$, then $-\infty \notin \Phi_M^p(X \times X^{\diamond})$. In the sequel, we assume that M is a nonempty subset of $X \times X^{\diamond}$.

Proposition 5.2. *Let X be an Hadamard space, $M \subseteq X \times X^{\diamond}$ and $p \in X$. Then for each $(x, x^{\diamond}) \in X \times X^{\diamond}$ we have:*

$$\Phi_M^p(x, x^{\diamond}) = \pi_p(x, x^{\diamond}) - \inf_{(y,y^{\diamond}) \in M} \langle x^{\diamond} - y^{\diamond}, \overrightarrow{yx}\rangle.$$

Proof. By Definition 5.1, for each $(x, x^{\diamond}) \in X \times X^{\diamond}$,

$$\begin{aligned}\Phi_M^p(x, x^{\diamond}) &= \sup_{(y,y^{\diamond}) \in M} \big\{\langle x^{\diamond}, \overrightarrow{py}\rangle - \langle y^{\diamond}, \overrightarrow{xy}\rangle\big\}\\ &= \sup_{(y,y^{\diamond}) \in M} \big\{\langle x^{\diamond}, \overrightarrow{px} + \overrightarrow{xy}\rangle - \langle y^{\diamond}, \overrightarrow{xy}\rangle\big\}\\ &= \langle x^{\diamond}, \overrightarrow{px}\rangle + \sup_{(y,y^{\diamond}) \in M} \langle x^{\diamond} - y^{\diamond}, \overrightarrow{xy}\rangle\\ &= \pi_p(x, x^{\diamond}) - \inf_{(y,y^{\diamond}) \in M} \langle x^{\diamond} - y^{\diamond}, \overrightarrow{yx}\rangle,\end{aligned}$$

we are done. $\square$

Proposition 5.3. *Let X be an Hadamard space, $M \subseteq X \times X^\diamond$ and $p \in X$. Then*

(i) $M^\mu = \{\Phi_M^p \leq \pi_p\}$.

(ii) If M is monotone, then $M \subseteq \{\Phi_M^p = \pi_p\}$.

(iii) If M is maximal monotone, then $X \times X^\diamond \setminus M \subseteq \{\Phi_M^p > \pi_p\}$; i.e., $\{\Phi_M^p \leq \pi_p\} \subseteq M$.

(iv) If M is maximal monotone, then $\{\Phi_M^p \geq \pi_p\} = X \times X^\diamond$.

(v) If M is maximal monotone, then $\{\Phi_M^p = \pi_p\} = M$.

(vi) If $\{\Phi_M^p = \pi_p\} = M$ and $\{\Phi_M^p \geq \pi_p\} = X \times X^\diamond$, then M is maximal monotone.

Proof. (i): By definition of M^μ, Proposition 5.2 and the definition of Φ_M^p, we have:

$$\begin{aligned}(x, x^\diamond) \in M^\mu &\Leftrightarrow \forall (y, y^\diamond) \in M,\ \langle x^\diamond - y^\diamond, \overrightarrow{yx} \rangle \geq 0 \\ &\Leftrightarrow \inf_{(y,y^\diamond)\in M} \langle x^\diamond - y^\diamond, \overrightarrow{yx} \rangle \geq 0 \\ &\Leftrightarrow \pi_p(x, x^\diamond) - \inf_{(y,y^\diamond)\in M} \langle x^\diamond - y^\diamond, \overrightarrow{yx} \rangle \leq \pi_p(x, x^\diamond) \\ &\Leftrightarrow \Phi_M^p(x, x^\diamond) \leq \pi_p(x, x^\diamond) \\ &\Leftrightarrow (x, x^\diamond) \in \{\Phi_M^p \leq \pi_p\}.\end{aligned}$$

(ii): Let $(x, x^\diamond) \in M$. By monotonicity of M we get:

$$\inf_{(y,y^\diamond)\in M} \langle x^\diamond - y^\diamond, \overrightarrow{yx} \rangle = 0.$$

Now, Proposition 5.2 completes the proof.

(iii): Let $(x, x^\diamond) \in X \times X^\diamond \setminus M$. By maximality of M, we know that $(x, x^\diamond)$ is not monotonically related to M. Hence there exists $(z, z^\diamond) \in M$ such that $\langle x^\diamond - z^\diamond, \overrightarrow{zx} \rangle < 0$, which implies that $-\inf_{(y,y^\diamond)\in M} \langle x^\diamond - y^\diamond, \overrightarrow{yx} \rangle > 0$. Now, it follows from Proposition 5.2 that

$$\Phi_M^p(x, x^\diamond) = \pi_p(x, x^\diamond) - \inf_{(y,y^\diamond)\in M} \langle x^\diamond - y^\diamond, \overrightarrow{yx} \rangle > \pi_p(x, x^\diamond).$$

Therefore, $X \times X^\diamond \setminus M \subseteq \{\Phi_M^p > \pi_p\}$ and so $\{\Phi_M^p \leq \pi_p\} \subseteq M$.

(iv): It follows from (ii) and (iii) that

$$X \times X^{\diamond} = M \cup (X \times X^{\diamond} \setminus M) \subseteq \{\Phi_M^p = \pi_p\} \cup \{\Phi_M^p > \pi_p\} = \{\Phi_M^p \geq \pi_p\};$$

i.e., $\{\Phi_M^p \geq \pi_p\} = X \times X^{\diamond}$.

(v): By using (ii) we conclude that $M \subseteq \{\Phi_M^p = \pi_p\}$. On the other hand, parts (iii) and (iv) imply that

$$\begin{aligned}\{\Phi_M^p = \pi_p\} &= \{\Phi_M^p \leq \pi_p\} \cap \{\Phi_M^p \geq \pi_p\} \\ &\subseteq M \cap (X \times X^{\diamond}) \\ &= M.\end{aligned}$$

Hence, $\{\Phi_M^p = \pi_p\} = M$.

(vi): Let $(x, x^{\diamond}) \in M$. By assumption $(x, x^{\diamond}) \in \{\Phi_M^p = \pi_p\}$, so

$$\inf_{(y,y^{\diamond}) \in M} \langle x^{\diamond} - y^{\diamond}, \overrightarrow{yx} \rangle = 0.$$

Therefore, for each $(y, y^{\diamond}) \in M$, we have $\langle x^{\diamond} - y^{\diamond}, \overrightarrow{yx} \rangle \geq 0$; i.e., M is monotone. It follows from part (i) and our assumptions that

$$M^{\mu} = M^{\mu} \cap (X \times X^{\diamond}) = \{\Phi_M^p \leq \pi_p\} \cap \{\Phi_M^p \geq \pi_p\} = \{\Phi_M^p = \pi_p\} = M.$$

Now, Proposition 4.4(ii) completes the proof.

Proposition 5.4. *Let X be an Hadamard space, $M \subseteq X \times X^{\diamond}$ and $p \in X$. Then*

(i) Φ_M^p is proper and l.s.c.

(ii) Φ_M^p is convex, if M has $\mathcal{F}_l$-property. In this case, $\Phi_M^p \in \Gamma(X \times X^{\diamond})$.

Proof. (i): Clearly Φ_M^p is proper. For each $(y, y^{\diamond}) \in X \times X^{\diamond}$, we define $\psi_{(y,y^{\diamond})}(x, x^{\diamond}) : X \times X^{\diamond} \to \mathbb{R}$ by $\psi_{(y,y^{\diamond})}(x, x^{\diamond}) = \pi_p(y, x^{\diamond}) + \pi_y(x, y^{\diamond})$. Let $\{(x_n, x_n^{\diamond})\} \subseteq X \times X^{\diamond}$ be such that $(x_n, x_n^{\diamond}) \to (x, x^{\diamond})$. By using Lemma 3.11, we get:

$$\pi_p(y, x_n^{\diamond}) + \pi_y(x_n, y^{\diamond}) \to \pi_p(y, x^{\diamond}) + \pi_y(x, y^{\diamond}),$$

or equivalently,

$$\psi_{(y,y^{\diamond})}(x_n, x_n^{\diamond}) \to \psi_{(y,y^{\diamond})}(x, x^{\diamond}),$$

which implies that $\psi_{(y,y^{\diamond})}$ is continuous. Consequently, Φ_M^p is lower semi-continuous, since $\Phi_M^p(x, x^{\diamond}) = \sup_{(y,y^{\diamond}) \in M} \psi_{(y,y^{\diamond})}(x, x^{\diamond})$.

(ii) Let $a, b \in X$, $x^\diamond \in X^\diamond$ and $\lambda \in [0,1]$. Again, by using $\psi_{(y,y^\diamond)}$ whenever $(y, y^\diamond) \in X \times X^\diamond$ and Lemma 3.12(i) we obtain:

$$\begin{aligned}\psi_{(y,y^\diamond)}((1-\lambda)a \oplus \lambda b, x^\diamond) &= \pi_p(y, x^\diamond) + \pi_y((1-\lambda)a \oplus \lambda b, y^\diamond)\\ &\leq \pi_p(y, x^\diamond) + (1-\lambda)\pi_y(a, y^\diamond) + \lambda \pi_y(b, y^\diamond)\\ &= (1-\lambda)(\pi_p(y, x^\diamond) + \pi_y(a, y^\diamond))\\ &\quad + \lambda(\pi_p(y, x^\diamond) + \pi_y(b, y^\diamond))\\ &= (1-\lambda)\psi_{(y,y^\diamond)}(a, x^\diamond) + \lambda\psi_{(y,y^\diamond)}(b, x^\diamond),\end{aligned}$$

which implies that $\psi_{(y,y^\diamond)}$ is convex. Thus

$$\Phi_M^p(x, x^\diamond) = \sup_{(y,y^\diamond)\in M} \psi_{(y,y^\diamond)}(x, x^\diamond),$$

is convex. Finally, by using (i), we obtain that $\Phi_M^p \in \Gamma(X \times X^\diamond)$.

Proposition 5.5. *Let X be an Hadamard space, $M \subseteq X \times X^\diamond$ be a monotone relation with $\mathcal{F}_l$-property and $p \in X$. Then there exists $h \in \Gamma(X \times X^\diamond)$ such that $\{h \geq \pi_p\} = X \times X^\diamond$ and $M \subseteq \{h = \pi_p\}$; in other words:*

(i) $h(x, x^\diamond) \geq \pi_p(x, x^\diamond)$, *for each* $(x, x^\diamond) \in X \times X^\diamond$.

(ii) $(x, x^\diamond) \in M \Rightarrow h(x, x^\diamond) = \pi_p(x, x^\diamond)$.

Proof. By using Remark 4.3 there exists $\widetilde{M} \in \mathfrak{M}(X)$ such that $M \subseteq \widetilde{M}$. Set $h := \Phi^p_{\widetilde{M}}$. It follows from Proposition 5.4 that $h \in \Gamma(X \times X^\diamond)$. Finally, (i) and (ii) follow from parts (iv) and (ii) of Proposition 5.3, respectively. □

Proposition 5.6. *Let X be an Hadamard space and let $M \subseteq X \times X^\diamond$ has $\mathcal{F}_g$-property. If there exists $h \in \Gamma(X \times X^\diamond)$ such that $\{h \geq \pi_p\} = X \times X^\diamond$ and $M \subseteq \{h = \pi_p\}$, then M is monotone.*

Proof. Take $(x, x^\diamond) \in M$ and $(y, y^\diamond) \in M$. Then $h(x, x^\diamond) = \pi_p(x, x^\diamond)$ and $h(y, y^\diamond) = \pi_p(y, y^\diamond)$. Now, $\{h \geq \pi_p\} = X \times X^\diamond$, convexity of h and $\mathcal{F}_g$-property of M imply that

$$\begin{aligned}0 &\leq \frac{1}{2}h(x, x^\diamond) + \frac{1}{2}h(y, y^\diamond) - h\Big(\frac{1}{2}x \oplus \frac{1}{2}y, \frac{1}{2}x^\diamond + \frac{1}{2}y^\diamond\Big)\\ &\leq \frac{1}{2}\pi_p(x, x^\diamond) + \frac{1}{2}\pi_p(y, y^\diamond) - \pi_p\Big(\frac{1}{2}x \oplus \frac{1}{2}y, \frac{1}{2}x^\diamond + \frac{1}{2}y^\diamond\Big)\\ &= \frac{1}{2}\langle x^\diamond, \overrightarrow{px}\rangle + \frac{1}{2}\langle y^\diamond, \overrightarrow{py}\rangle - \Big\langle \frac{1}{2}x^\diamond + \frac{1}{2}y^\diamond, \overrightarrow{p\big(\tfrac{1}{2}x \oplus \tfrac{1}{2}y\big)}\Big\rangle\\ &= \frac{1}{2}\langle x^\diamond, \overrightarrow{px}\rangle + \frac{1}{2}\langle y^\diamond, \overrightarrow{py}\rangle - \frac{1}{2}\Big\langle x^\diamond, \overrightarrow{p\big(\tfrac{1}{2}x \oplus \tfrac{1}{2}y\big)}\Big\rangle - \frac{1}{2}\Big\langle y^\diamond, \overrightarrow{p\big(\tfrac{1}{2}x \oplus \tfrac{1}{2}y\big)}\Big\rangle\end{aligned}$$

$$\leq \frac{1}{2}\langle x^\diamond, \overrightarrow{px}\rangle + \frac{1}{2}\langle y^\diamond, \overrightarrow{py}\rangle - \frac{1}{4}\langle x^\diamond, \overrightarrow{px}\rangle - \frac{1}{4}\langle x^\diamond, \overrightarrow{py}\rangle - \frac{1}{4}\langle y^\diamond, \overrightarrow{px}\rangle - \frac{1}{4}\langle y^\diamond, \overrightarrow{py}\rangle$$
$$= \frac{1}{4}\big(\langle x^\diamond, \overrightarrow{px}\rangle - \langle x^\diamond, \overrightarrow{py}\rangle - \langle y^\diamond, \overrightarrow{px}\rangle + \langle y^\diamond, \overrightarrow{py}\rangle\big)$$
$$= \frac{1}{4}\langle x^\diamond - y^\diamond, \overrightarrow{yx}\rangle.$$

Thus M is monotone. □

Theorem 5.7. *Let X be an Hadamard space, $M \subseteq X \times X^\diamond$ has $\mathcal{F}$-property and $p \in X$. Then M is monotone if and only if there exists $h \in \Gamma(X \times X^\diamond)$ such that $\{h \geq \pi_p\} = X \times X^\diamond$ and $M \subseteq \{h = \pi_p\}$.*

Proof. It is an immediate consequence of Proposition 5.5 and Proposition 5.6 □

Corollary 5.8. *Let X be a flat Hadamard space. Then $M \subseteq X \times X^\diamond$ is a monotone relation if and only if there exists $h \in \Gamma(X \times X^\diamond)$ such that $\{h \geq \pi_p\} = X \times X^\diamond$ and $M \subseteq \{h = \pi_p\}$.*

Proof. It follows from Theorem 3.5 and Theorem 5.7. □

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